Bayesian Calculations: Simulation of Coin Flipping

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1 Background

Bayes Theorem allows us to state our *prior* beliefs, to calculate the *likelihood* of our data given those beliefs, and then to update those beliefs with data, thus arriving at a set of *posterior* beliefs. However, Bayesian calculations can be difficult to understand. This document attempts to provide a simple walkthrough of some Bayesian calculations.

```r
library(pander) # nice tables

library(tibble) # data frames

library(ggplot2) # beautiful graphs
```

2 Bayes Rule

Mathematically Bayes Theorem is as follows:

$$P(H \mid D) = \frac{P(D \mid H)P(H)}{P(D)}$$

In words, Bayes Theorem may be written as follows:

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{data}}$$

Our posterior beliefs are proportional to our prior beliefs, multiplied by the likelihood of those beliefs, given the data.

3 This Example

In this example, we provide an example of using Bayes Theorem to examine our conclusions about the proportion of heads when a coin is flipped 10 times.

Conventionally, we call this proportion that we are trying to estimate θ.

For the sake of simplicity, this example uses a relatively simple set of prior beliefs about 3 possible values for the proportion θ.

R code in this example is adapted and simplified from Kruschke (2011), p. 70

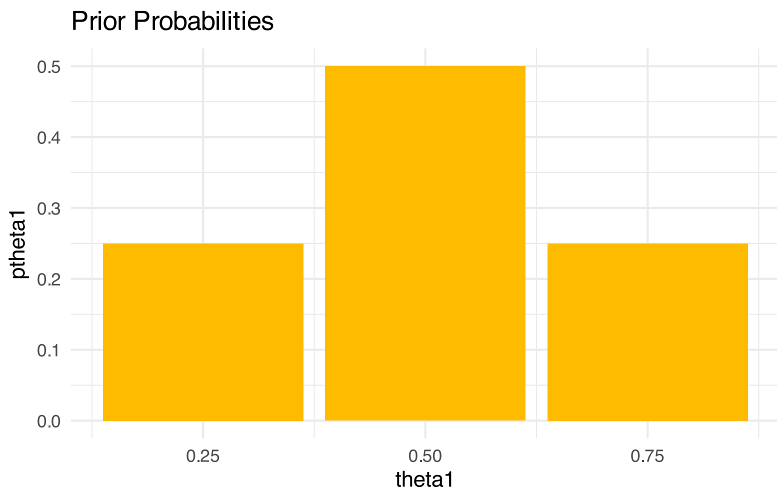
4 Prior

We set a simple set of prior beliefs, concerning 3 values of θ, the proportion of heads.

```r
theta1 <- c(.25, .50, .75) # candidate parameter values

ptheta1 <- c(.25, .50, .25) # prior probabilities

ptheta1 <- ptheta1/sum(ptheta1) # normalize
```

Our values of θ are *0.25*, *0.5* and *0.75*, with probabilities $P(\theta)$ of *0.25*, *0.5* and *0.25*.

```r
ggplot(data = NULL,
       aes(x = theta1,
           y = ptheta1)) +
  geom_bar(stat = "identity",
           fill = "#FFBB00") +
  labs(title = "Prior Probabilities") +
  theme_minimal()
```




```
myBayesianEstimates <- tibble(theta1, ptheta1) # data frame

pander(myBayesianEstimates) # nice table
```

theta1	ptheta1
0.25	0.25
0.5	0.5
0.75	0.25

5 The Data

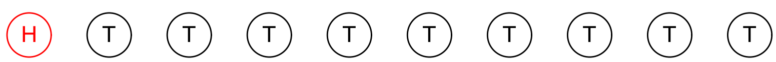
10 coin flips. 1 Heads. 9 Tails.

```
data1 <- c(1, 0, 0, 0, 0, 0, 0, 0, 0, 0) # the data

data1_factor <- factor(data1,
                       levels = c(0, 1),
                       labels = c("T", "H"))
```

```
n_heads <- sum(data1 == 1) # number of heads

n_tails <- sum(data1 == 0) # number of tails
```

```
x <- seq(1, 10) # x goes from 1 to 10

y <- rep(1, 10) # y is a sequence of 10 1's
```

```
coindata <- data.frame(x, y, data1_factor) # data for visualization

ggplot(coindata,
       aes(x = x,
           y = y,
           label = data1_factor,
           color = data1_factor)) +
  geom_point(size = 10, pch = 1) +
  geom_text() +
  labs(x = "",
       y = "") +
  scale_color_manual(values = c("black", "red")) +
  theme_void() +
  theme(legend.position = "none")
```



6 Likelihood

The likelihood is the probability that a given value of θ would produce this number of heads.

The probability of multiple independent events A, B, C, etc. is $P(A, B, C, ...) = P(A) \times P(B) \times P(C) \times ...$.

Therefore, in this case, the likelihood is proportional to $[P(heads)]^{\text{number of heads}}$ and multiply this by $[P(tails)]^{\text{number of tails}}$.

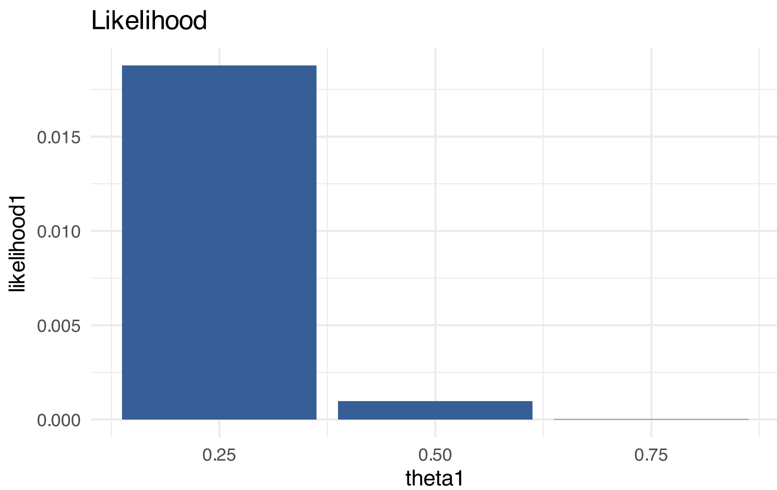
Thus:

$$\mathcal{L}(\theta) \propto \theta^{\text{number of heads}} \times (1 - \theta)^{\text{number of tails}}$$

```
likelihood1 <- theta1^n_heads * (1 - theta1)^n_tails # likelihood
```

```
ggplot(data = NULL,
       aes(x = theta1,
           y = likelihood1)) +
  geom_bar(stat = "identity",
           fill = "#375E97") +
```

```r
  labs(title = "Likelihood") +
  theme_minimal()
```



At this point our estimates include not only a value of θ and $P(\theta)$, but also the likelihood, $\mathcal{L}(\theta)$.

```r
myBayesianEstimates <- tibble(theta1, ptheta1, likelihood1)

pander(myBayesianEstimates) # nice table
```

theta1	ptheta1	likelihood1
0.25	0.25	0.01877
0.5	0.5	0.0009766
0.75	0.25	2.861e-06

7 Posterior

We then calculate the denominator of Bayes theorem:

$$\Sigma[\mathcal{L}(\theta) \times P(\theta)]$$

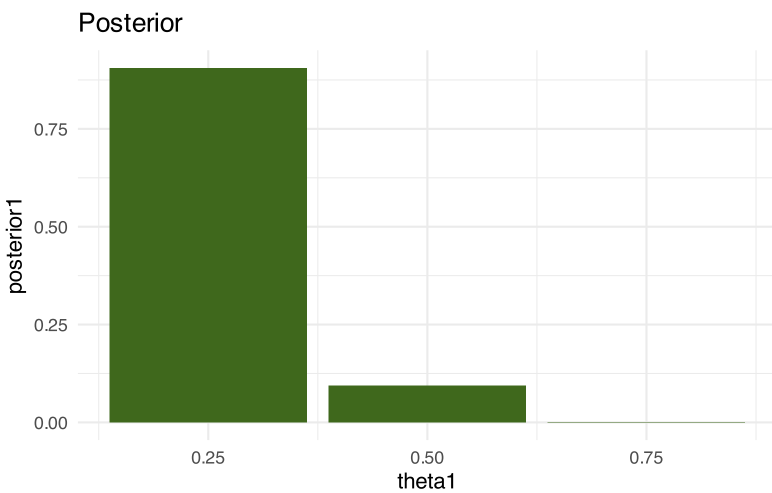
```r
pdata1 <- sum(likelihood1 * ptheta1) # normalize
```

We then use Bayes Rule to calculate the posterior:

$$P(H \mid D) = \frac{P(D \mid H)P(H)}{P(D)}$$

```r
posterior1 <- likelihood1 * ptheta1 / pdata1 # Bayes Rule
```

```r
ggplot(data = NULL,
       aes(x = theta1,
           y = posterior1)) +
  geom_bar(stat = "identity",
           fill = "#3F681C") +
  labs(title = "Posterior") +
  theme_minimal()
```



Our estimates now include θ, $P(\theta)$, $\mathcal{L}(\theta)$ and $P(\theta \mid D)$.

```r
myBayesianEstimates <- tibble(theta1, ptheta1, likelihood1, posterior1)

pander(myBayesianEstimates) # nice table
```

theta1	ptheta1	likelihood1	posterior1
0.25	0.25	0.01877	0.9056
0.5	0.5	0.0009766	0.09423
0.75	0.25	2.861e-06	0.000138

Estimates Are A Combination of Prior and Likelihood

Notice how $\theta = .5$ has the highest prior probability. $\theta = .25$ has the highest likelihood. $\theta = .75$ has an equivalent prior probability to $\theta = .25$ but a much lower likelihood. The posterior is proportional to the prior multiplied by the likelihood. In this case the posterior estimates strongly favor $\theta = .25$.

8 Credits

Prepared by Andy Grogan-Kaylor agrogan@umich.edu, https://agrogan1.github.io/.

Questions, comments and corrections are most welcome.